On the generation of surface waves by shear flows. Part 4

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The transfer of energy from wind to short surface waves through the viscous Reynolds stress in the immediate neighbourhood of the surface is explored. Resonance between the Tollmien–Schlichting waves for a given wind profile and the free-surface waves is shown to be an important possibility. Numerical results are given for water wave generation by a wind having a velocity profile that is linear in the neighbourhood of the surface and asymptotically logarithmic.

1. Introduction

Four, essentially different, mechanisms for the transfer of energy from wind to surface waves have recently been proposed and analysed.

(*a*) Phillips (1957), assuming no interaction between surface waves and wind, has considered the direct action of turbulent fluctuations in pressure on the free surface.

(*b*) Miles (1957*a*, 1959*a*) and Brooke Benjamin (1959) have considered energy transfer through the *inviscid Reynolds stress* in the critical layer (where wave speed equals mean wind speed) of a *curved* velocity profile on the assumption that perturbations of the turbulent Reynolds stresses are negligible.

(*c*) Brooke Benjamin (1959) and Longuet-Higgins (1952, unpublished) have considered energy transfer through the *viscous Reynolds stress* in the immediate neighbourhood of the surface.‡

(*d*) Miles (1959*b*) has considered *static* or Kelvin–Helmholtz instability, modifying Kelvin's original model to allow for variation of mean wind speed with distance from the surface.

We first remark that turbulent fluctuations (in so far as they exist) are bound to play some role in the initial excitation of all surface waves, so that the real question for (*a*) is whether such a mechanism can account for the complete energy transfer or whether it is augmented by interaction between surface waves and wind (see Miles 1960*a*). Presently available data, although rather inadequate for any firm decisions, appear to indicate that both (*a*) and (*b*) are important for the formation of the longer gravity waves on deep water and that (*d*) is not. It appears fairly certain, on the other hand, that (*d*) is important for the formation

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‡ It must be emphasized that, in first approximation, the energy transfer at the interface is through the out-of-phase component of the pressure and that energy transfer through the shear stress enters only in the next approximation. It was, among other things, to emphasize this distinction that the term *viscous Reynolds stress*, rather than *viscous stress*, was selected and has been retained.

of waves on viscous liquids such as oil, the quantitative agreement between theory and experiment being excellent.

None of (*a*), (*b*) or (*d*) appears adequate to explain the formation of relatively short waves on initially smooth water (it seems quite unlikely that any linear theory could prove adequate to explain their formation on rough water), and the aim of the following analysis is to explore the possible importance of (*c*) for capillary and short gravity waves. We remark that this mechanism may be especially important in the formation of waves on rather shallow water, where bottom friction inhibits the formation of longer waves (Jeffreys 1926). It must be emphasized, however, that we shall consider only those waves that move downstream relative to the surface current and that we shall neglect the mean flow in the lower fluid. Waves moving downstream (relative to the bottom) more slowly than the surface current must obtain their energy from the mean flow in the lower fluid, probably through the viscous Reynolds stress at the bottom (Miles 1960*b*); however, this mechanism is likely to be important only for very thin films.

The principal reasons for suspecting that turbulent fluctuations, acting alone, are inadequate for the generation of short water waves on initially smooth water are that the fluctuations at such small wavelengths are relatively weak and that they are convected downstream too rapidly to account (through the 'resonance' mechanism proposed by Phillips) for the *straight-crested* waves that are observed. There can be little question, on the other hand, that turbulent fluctuations are responsible for the minute, random agitations of any body of water.

We may rule out energy transfer associated with profile curvature to capillary waves through the consideration that this energy transfer increases roughly as the square of the wind speed *for a fixed ratio of wave speed to wind speed*, whereas laminar dissipation in the water increases as the cube of the wind speed (in contrast with gravity waves, where it varies inversely as the wind speed). The two would be in equilibrium if (Miles 1957*a*)

$$\beta \rho_a U^2 = 4\mu_w kc, \tag{1.1}$$

where β denotes the energy-transfer coefficient, referred to a characteristic velocity U, ρ_a the air density, μ_w the viscosity of the water, k the wave-number, and c the wave speed. Assuming the logarithmic profile $U(y) = U \log(y/z_0)$, the maximum value of β is 3·4 (Miles 1959*a*). The minimum values of k and c for capillary waves are $2\pi/1{\cdot}7\,\mathrm{cm}^{-1}$ and 23 cm/sec, so that (1.1) implies that U must exceed 29 cm/sec in order to overcome laminar dissipation in the water (assuming $\rho_a = 1{\cdot}2 \times 10^{-3}$ and $\mu_w = 10^{-2}$ in c.g.s. units). But if we rewrite (1.1) in the form

$$U = \frac{1}{4}\left(\frac{\rho_a}{\mu_w}\right)\left(\frac{c^2}{k}\right)\frac{\beta}{(c/U)^3}, \tag{1.2}$$

and note that c^2/k cannot exceed 150 (c.g.s. units) for capillary waves and that c/U must exceed 3 if the profile curvature at the critical layer is to be appreciable, we find that U cannot exceed 0·6 cm/sec. It follows that the equilibrium implied by (1.1) cannot be attained and hence that laminar dissipation necessarily exceeds the energy transfer associated with profile curvature.

In contrast to the lower bound placed on wave speed by the requirement that the critical layer be at a sufficient elevation to guarantee appreciable profile curvature, the energy transfer associated with the viscous Reynolds stress imposes the upper bound (Brooke Benjamin 1959, (7.43) ff.)

$$c < 2{\cdot}3(U_*/k\nu_a)^{\frac{1}{3}}\,U_*, \tag{1.3}$$

where U_* denotes the friction velocity for the profile and ν_a the kinematic viscosity for the air. This suggests that the inviscid and viscous Reynolds stresses are likely to play complementary roles in wave generation, and that the latter is dominant for short waves.

We shall proceed to investigate this possibility by first (in § 2) obtaining the equations of two-dimensional wave motion in a slightly viscous liquid of finite depth that is subjected to prescribed stresses at its surface. We then shall calculate the surface stresses produced by a parallel shear flow above the liquid along the lines laid down by Brooke Benjamin (1959), whose work is recapitulated in a slightly revised form in §§ 3 and 4. This formulation leads to the Orr–Sommerfeld equation, the asymptotic solution to which can be expressed in terms of Airy integrals and the solution to the inviscid problem. Methods appropriate to the solution of the latter boundary-value problem have been discussed in a separate paper (Miles 1962), the results of which are recapitulated in § 5.

The development of §§ 2–5 culminates in an eigenvalue equation for the complex wave speed c. Assuming the specific gravity of the upper fluid to be small, the solutions to this equation comprise: (1) the free-surface waves of the lower fluid, perturbed by the shear flow in the upper fluid, and (ii) the Tollmien–Schlichting waves of the upper fluid, perturbed (relative to the waves for the same shear flow over a rigid wall) by the motion of the lower fluid. The waves of class (i) are of the same general class as those considered in the aforementioned references (Brooke Benjamin 1959; Miles 1957*a*, 1959*a*, *b*, 1960*a*); those of class (ii) have been considered previously by Brooke Benjamin (1960) and Betchov (1961) in connexion with the drag of a flexible body—e.g. a porpoise.† These two classes of waves are relatively independent throughout most of the wind speed, wavelength spectrum, but resonance between them appears to afford an important and interesting possibility for a sharply peaked energy transfer from shear flow to surface wave over a limited portion of this spectrum.

The growth of waves of class (i) is investigated in more detail in §§ 7 and 8, and numerical results are presented for the generation of water waves by a shear flow having a profile corresponding to the mean flow in a turbulent boundary layer (linear at surface and asymptotically logarithmic). These results suggest that energy transfer from wind to waves through the action of the viscous Reynolds stress may be of considerable practical importance.

2. Surface waves on a slightly viscous liquid

We consider a liquid of density ρ, kinematic viscosity ν_w and surface tension $\rho\sigma$ that is bounded below by the rigid plane $y = -d$ and above by the surface wave

$$y = y_0(x,t) = a\,e^{ik(x-ct)} \quad (ka \ll 1), \tag{2.1}$$

† See also Becker (1960).

in the Cartesian co-ordinates x and y. We require the equation of motion for this wave under the action of the normal (positive into the liquid) stress p_0 and the tangential stress τ_0, which stresses we assume to be generated aerodynamically by a light fluid of density $s\rho$ and kinematic viscosity ν_a and to be expressed in the form

$$(p_0, \tau_0) = s\rho(P, T)\, y_0. \tag{2.2}$$

We also shall pose the restrictions

$$s \ll 1, \quad \frac{s\nu_a}{\nu_w} \ll 1, \quad R_w = \frac{c}{k\nu_w} \gg 1, \quad \frac{cd}{\nu_w} \equiv kdR_w \gg 1, \qquad (2.3a, b, c, d)$$

where R_w is an appropriate Reynolds number for the wave motion in the liquid. The restriction (2.3b) justifies the neglect of the shear flow in the liquid, but we note that this approximation requires reconsideration for very thin films (see Miles 1960b).

We may derive u and v, the x- and y-components of the velocity in the liquid, from a stream function $\psi(y)\, y_0(x,t)$ according to

$$u = \psi'(y)\, y_0(x,t), \quad v = -ik\psi(y)\, y_0(x,t), \qquad (2.4a, b)$$

where ψ satisfies

$$(D^2 - k^2)(D^2 - m^2)\psi = 0, \quad m^2 = k^2 - i(kc/\nu_w), \qquad (2.5a, b)$$

and $D\psi \equiv \psi'(y)$. The kinematic boundary conditions on ψ are

$$\psi = \psi' = 0 \quad \text{at} \quad y = -d \quad \text{and} \quad \psi(0) = c, \qquad (2.6a, b, c)$$

where we have invoked the last condition at $y = 0$, rather than $y = y_0$, by virtue of the restriction $ka \ll 1$. The dynamic boundary conditions (again invoking $ka \ll 1$) are

$$-p^{(yy)} \equiv \rho[c\psi' + (\nu_w/ik)(\psi''' - 3k^2\psi')]\, y_0$$
$$= \rho(g + \sigma k^2)\, y_0 + p_0 \tag{2.7}$$

and

$$p^{(xy)} \equiv \rho\nu_w(\psi'' + k^2\psi)\, y_0 = \tau_0, \tag{2.8}$$

where $p^{(xy)}$ and $p^{(yy)}$ are components of the Cartesian stress tensor (cf. Lamb 1945, §349).

Neglecting terms of $O(e^{-md})$ by virtue of the restriction (2.3d), a solution to (2.5) that satisfies the kinematic boundary conditions (2.6) is given by

$$\psi = (c - A)\,[C - (m/k)\, S]^{-1}\{\cosh[k(y+d)]$$
$$- (m/k)\sinh[k(y+d)] - e^{-m(y+d)}\} + A\, e^{my}, \tag{2.9}$$

where A is an arbitrary constant, $\mathscr{R}\{m\} > 0$, and

$$C = \cosh kd, \quad S = \sinh kd. \tag{2.10}$$

We remark that the solutions $e^{-m(y+d)}$ and e^{my} are significant only in thin boundary layers at $y = -d$ and $y = 0$, respectively.

Considering next the determination of A, we substitute τ_0 from (2.2) and ψ from (2.9) into (2.8) to obtain

$$A = -2ik\nu_w + is(kc)^{-1}T. \tag{2.11}$$

Substituting (2.9) and (2.11) into (2.7), we then obtain the secular equation for the determination of c as a function of k:

$$k\left(\frac{mC-kS}{mS-kC}\right)(c+2ik\nu_w)^2+(2k\nu_w)^2 m$$
$$= g+\sigma k^2+s[P+iT\coth kd+O(R_w^{-\frac{1}{2}})]. \qquad (2.12)$$

Up to this point, the left-hand side of (2.12) is accurate (as $ka \to 0$) within an error factor $1+O(e^{-md})$. If we now retain only the dominant terms of algebraic order in R_w as $R_w \to \infty$, we may neglect the term $(2k\nu_w)^2 m$ and let

$$\frac{mC-kS}{mS-kC} = \coth kd+\left(\frac{k}{m}\right)\operatorname{csch}^2 kd+\left(\frac{k}{m}\right)^2\coth kd\operatorname{csch}^2 kd+\ldots$$
$$= \coth kd+\left(\frac{i}{R_w}\right)^{\frac{1}{2}}\operatorname{csch}^2 kd+O(R_w^{-1}). \qquad (2.13)$$

Introducing the abbreviation (for the speed of undamped, deep-water waves)

$$c_0 = (gk^{-1}+\sigma k)^{\frac{1}{2}}, \qquad (2.14)$$

together with the approximation (2.13), we may transform (2.12) to

$$(c+2ik\nu_w)^2-c_0^2[\tanh kd-(1+i)(2R_w)^{-\frac{1}{2}}\operatorname{sech}^2 kd] = sk^{-1}[P\tanh kd+iT]. \qquad (2.15)$$

Finally, we may reduce (2.15) to

$$(c+2ik\nu_w)^2-c_0^2[1-i(8/R_w)^{\frac{1}{2}}e^{-2kd}] = sk^{-1}(P+iT) \qquad (2.16)$$

for deep-water waves (say $kd > 2$).

We observe that the damping in the boundary layer at $y = -d$ is of lower order in R_w than that in the boundary layer at $y = 0$ ($R_w^{-\frac{1}{2}}$ *vs* R_w^{-1}) but falls off exponentially with kd, and it is for the latter reason that we have not neglected $2ik\nu_w$ relative to c in (2.15) and (2.16). The contributions of these two boundary layers to the left-hand sides of (2.15) and (2.16) are equal, within the approximations of (2.3c, d), if $d = D$, where

$$D = (4k)^{-1}\log(|c|/2k\nu_w), \qquad (2.17)$$

and we may neglect the term of $O(e^{-2kd})$ in (2.16) if $d \gg D$. (We have already neglected the real term of this order by virtue of the less stringent restriction $kd \gg 1$.)

We shall require the tangential velocity at $y = y_0$ in the form

$$u = \psi_0' y_0. \qquad (2.18)$$

Making use of (2.4a), (2.9), (2.11) and (2.13), we obtain

$$\psi_0' = kc\coth kd[1+O(R_w^{-\frac{1}{2}})]. \qquad (2.19)$$

We emphasize that the *tangential* and *normal* components of velocity in $y < y_0$ are equivalent to u and $-v$, respectively, within an error factor $1+O(ka)$. This is not generally true in $y > y_0$ (see § 3 below) because of the shear flow there.

3. Aerodynamic boundary-value problem

Brooke Benjamin's (1959) formulation of the equation governing small pertubations relative to the parallel shear flow $U(y)$ in $y > y_0$ leads to the following boundary-value problem for the quasi-stream-function $F(\eta)$, where η is a curvilinear co-ordinate defined such that $y = y_0(x,t)$ transforms to $\eta = 0$ within a factor $1+O(ka)$:†

$$(U-c)(F''-k^2F)-U''F = (\nu_a/ik)[F^{\text{iv}}-2k^2F'' + k^4F+(U^{\text{iv}}-2kU''')e^{-k\eta}], \tag{3.1}$$

$$F_0 = c, \quad F_0' = \psi_0'-U_0'. \tag{3.2}$$

Here and subsequently, the subscript zero denotes evaluation at $\eta = 0$, $U \equiv U(\eta)$, and $\psi_0' y_0$ denotes the tangential velocity at $\eta = 0$ (cf. (2.18) and (2.19)). We also introduce the subscript c to denote evaluation at the critical layer $\eta = \eta_c$, where

$$U(\eta_c) = c. \tag{3.3}$$

An appropriate characteristic length for the critical layer is

$$\delta = (\nu_a/U_c'k)^{\frac{1}{3}}, \tag{3.4}$$

and if the parameter

$$\epsilon = k\delta = (\nu_a k^2/U_c')^{\frac{1}{3}} \tag{3.5}$$

is sufficiently small the right-hand side of (3.1) is important only in the neighbourhoods $\eta = \eta_c + O(\delta)$ and $\eta = O(\delta)$. Moreover, previous studies of energy transfer from shear flows to surface waves (Brooke Benjamin 1959; Miles 1959*a*) have indicated that aerodynamic viscous effects are likely to be significant only if $\eta_c = O(\delta)$, in which case we need not distinguish between the neighbourhoods of $\eta = \eta_c$ and $\eta = 0$ in considering these effects. The asymptotic solution to (3.1) as $\epsilon \to 0$ then is given by (Lin 1955, §§3.4 and 3.6)

$$F(\eta) = \phi(\eta)+f(\eta), \tag{3.6}$$

where $\phi(\eta)$ satisfies the inviscid Orr–Sommerfeld equation

$$(U-c)(\phi''-k^2\phi)-U''\phi = 0, \tag{3.7}$$

the viscous solution $f(\eta)$ is given by

$$f(\eta) = C\int_{\infty}^{(\eta/\delta)-z} d\zeta \int_{\infty}^{\zeta} H_{\frac{1}{3}}^{(1)}[\tfrac{2}{3}(i\zeta')^{\frac{3}{2}}]\,d\zeta', \tag{3.8}$$

C is an undetermined constant, and

$$z = \eta_c/\delta \doteqdot c/U_c'\delta \tag{3.9}$$

is an appropriate measure of the wave speed c for the viscous solution. We also introduce the parameter

$$\gamma = -cU_c''/U_c'^2 \tag{3.10}$$

† Brooke Benjamin's formulation was dimensionless, but otherwise (3.1) is identical with his (3.1). A sign error in his (2.9) has been corrected and β replaced by ψ_0' to obtain (3.2) above.

as an appropriate measure of profile curvature and remark that the approximation (3.8) to the viscous solution $f(\eta)$ is valid only for $\gamma \ll 1$. Lin (1955, §8.7) has shown how (3.8) may be rendered valid for non-small values of γ through a re-definition of z, but this extension does not appear to be worthwhile for the applications contemplated herein.

4. Surface stresses

Neglecting terms of $O(\epsilon^2)$ compared with unity (as is consistent with the asymptotic approximation of the preceding section), we may approximate the normal stress at $y = y_0$ by the aerodynamic pressure and calculate the parameters of (2.2) according to (Brooke Benjamin 1959)†

$$P = U_0'\phi_0 + c\phi_0' \tag{4.1}$$

and
$$T/P = ikf_0''/f_0'''. \tag{4.2}$$

Then, introducing

$$w(k,c) = [1+(U_0'/c)(\phi_0/\phi_0')]^{-1}, \tag{4.3}$$

$$\mathscr{F}(z) = [1+(U_0'/c)(f_0/f_0')]^{-1}, \tag{4.4}$$

and
$$G(z) = f_0''/i\delta f_0''', \tag{4.5}$$

and invoking the boundary conditions (3.2) on the solution (3.6), we may solve for P and T in the forms

$$P = c\left[\frac{U_0' + \psi_0'(\mathscr{F}-1)}{\mathscr{F}-w}\right], \tag{4.6}$$

and
$$T = -\epsilon GP. \tag{4.7}$$

The function $\mathscr{F}(z)$ has been plotted and tabulated for $z = 1{\cdot}0\,(0{\cdot}2)\,5{\cdot}0$ by Lin (1955, p. 41) and for $z = -6{\cdot}0\ (0{\cdot}1)\ 10{\cdot}0$ by Miles (1960*b*). Approximations for small and large z are

$$\mathscr{F}(z) = 1{\cdot}288\,e^{-5i\pi/6}z + 0{\cdot}686\,e^{-2i\pi/3}z^2 + O(z^3) \tag{4.8a}$$

and
$$\sim 1 + e^{i\pi/4}z^{-\frac{3}{2}} + \tfrac{9}{4}iz^{-3} + O(z^{-\frac{9}{2}}). \tag{4.8b}$$

The function $G(z)$ may be expressed in terms of the Airy integral of the first kind according to

$$G(z) = e^{-2i\pi/3}Ai(ze^{-5i\pi/6})/Ai'(ze^{-5i\pi/6}) \tag{4.9a}$$

$$= \frac{1{\cdot}372\,e^{i\pi/3} + iz + 0{\cdot}229\,e^{-i\pi/6}z^3 + \frac{1}{12}z^4 + O(z^5)}{1 + 0{\cdot}686\,e^{-2i\pi/3}z^2 - \frac{1}{3}iz^3 + O(z^5)} \tag{4.9b}$$

and
$$\sim e^{3i\pi/4}z^{-\frac{1}{2}} - \tfrac{7}{48}z^{-2} + O(z^{-\frac{7}{2}}). \tag{4.9c}$$

The real and imaginary parts of G are plotted in figure 1.

† Equation (4.1) follows from Brooke Benjamin's (3.12) after identifying the latter as an invariant of the inviscid differential equation within the approximations already imposed. Equation (4.2) follows from Brooke Benjamin's (2.12) and (2.16) after invoking his boundary conditions (2.7) and (2.8) and neglecting terms of $O(\epsilon^2)$.

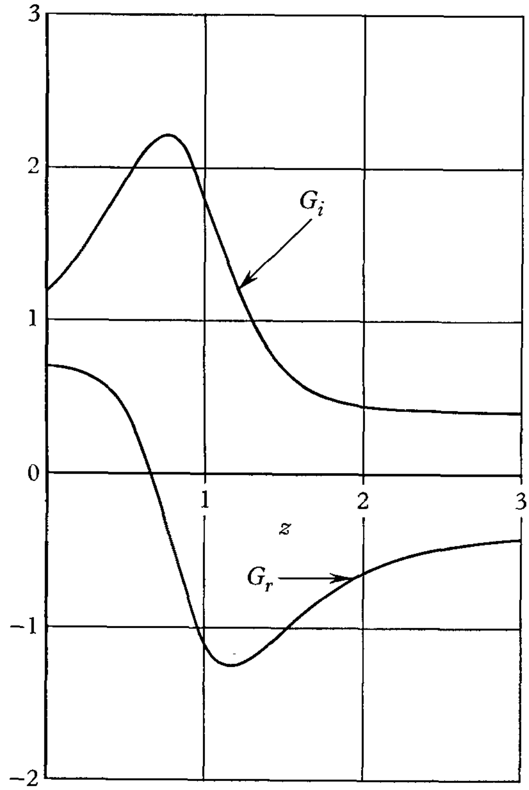

FIGURE 1. The real and imaginary parts of $G(z)$, as given by (4.9).

5. Inviscid solution

The well known method of Heisenberg (Lin 1955, §3.4) for the solution of the differential equation (3.7) is not generally suitable for the profiles of interest in the present context. A more powerful procedure is to introduce the new dependent variable

$$\Omega(\eta) = (U-c)^{-1}[U'\phi-(U-c)\phi']^{-1}\phi, \tag{5.1}$$

which satisfies the Riccati equation

$$\Omega' = k^2(U-c)^2\Omega^2-(U-c)^{-2}. \tag{5.2}$$

We then may show that (Miles 1962)

$$w = U_0'c\left[\Omega_1-\frac{1}{U_1'(U_1-c)}-\int_0^{\eta_1}\frac{U''d\eta}{U'^2(U-c)}\right][1+O(k\eta_1)^2], \tag{5.3}$$

where the subscript 1 implies evaluation at the point $\eta = \eta_1$ defined such that

$$k|\eta_c| < k\eta_1 \ll 1, \quad |U_1-c| \gg c_i, \tag{5.4a,b}$$

and the path of integration passes under $\eta = \eta_c$. This approximation implies (cf. Lin 1955, p. 37)

$$w_i = \pi(U_0'/U_c')\gamma[1+O(k|\eta_c|)^2]. \tag{5.5}$$

6. Eigenvalue problem

It now remains to determine the complex wave speed c as a function of the (real) wave-number k for prescribed values of the remaining parameters (s, ϵ, etc.). Combining (2.15), (2.19), (4.6) and (4.7), we obtain the eigenvalue equation

$$(c+2ik\nu_w)^2 - c_0^2[\tanh kd - (1+i)(2R_w)^{-\frac{1}{2}}\operatorname{sech}^2 kd] = \frac{sU_0'c}{k}\left[\frac{\tanh kd + \epsilon H(z) + O(\epsilon^2)}{\mathscr{F}(z) - w(k,c)}\right], \quad (6.1)$$

where (letting $U_c' \doteqdot U_0'$ in the coefficient of $\mathscr{F}-1$)

$$H(z) = z(\mathscr{F}-1) - iG \quad (6.2a)$$

$$= 1{\cdot}372\, e^{-i\pi/6} + 0{\cdot}35\, e^{-5i\pi/6} z^2 + O(z^3) \quad (6.2b)$$

and

$$\sim 2e^{i\pi/4} z^{-\frac{1}{2}} + O(z^{-2}). \quad (6.2c)$$

We remark that the term $\epsilon H(z)$ may be significant, even for small ϵ, if either d is not large compared with δ or the contribution of ϵH_i to the imaginary part of the right-hand side of (6.1) is comparable with the other contributions to that imaginary part. We also remark that, in accordance with (4.8a) and (5.3), both $\mathscr{F}$ and w vanish like c as $c \to 0$, in consequence of which the right-hand side of (6.1) is finite in this limit.

Let us consider first the limiting behaviour of c as $s \to 0$ with U_0' fixed. Then, either the left-hand side of (6.1) or $\mathscr{F}-w$ in the denominator of the right-hand side must vanish—say either $c = c_1$ or $c = c_2$, where (for waves running in the positive-x direction)

$$c_1 = c_0(\tanh kd)^{\frac{1}{2}} - (1+i)(k\nu_w c_0/2)^{\frac{1}{2}}(\tanh kd)^{\frac{1}{4}}\operatorname{csch} 2kd - 2ik\nu_w + O(R_w^{-\frac{3}{4}}, R_w^{-1}\operatorname{csch} 2kd), \quad (6.3)$$

and

$$\mathscr{F}(c_2/U_c'\delta) = w(k, c_2). \quad (6.4)$$

The wave speed c_1 corresponds to free-surface waves damped by viscous stresses in the boundary layers at $y = y_0$ and $y = -d$, which yield the contributions $-2ik\nu_w$ and $-(1+i)(k\nu_w c_0/2)^{\frac{1}{2}}(\tanh kd)^{\frac{1}{4}}\operatorname{csch} 2kd$, respectively. The wave speed c_2 corresponds to the Tollmien–Schlichting waves (not necessarily neutral) associated with small perturbations of the basic shear flow $U(y)$ over the rigid plane $y = 0$.

Now let us consider the first-order (in s) perturbations of c_1 and c_2 on the provisional hypothesis that those perturbations are small. The results are

$$c = c_1 + \frac{sU_0'}{2k}\left[\frac{\tanh kd + k\delta H(z)}{\mathscr{F}(z) - w(k,c)}\right]_{c=c_0(\tanh kd)^{\frac{1}{2}}} \quad (6.5)$$

and

$$c = c_2 + sU_0'\delta\left[\frac{\tanh kd + k\delta H(z)}{(\mathscr{F}'/U_0'\delta) - (\partial w/\partial c)}\right]_{c=c_2} \times \{(c_2 + 2ik\nu_w)^2 - c_0^2[\tanh kd - (1+i)(k\nu_w/2c_2)^{\frac{1}{2}}\operatorname{sech}^2 kd]\}^{-1} \quad (6.6)$$

within the approximations already imposed. The result (6.5) generalizes previous results of Brooke Benjamin (1959) and Miles (1957a, 1959a); (6.6) generalizes previous results of Brooke Benjamin (1960) and Betchov (1961).

The approximations (6.5) and (6.6) break down if either (*a*) the air speed is so high that sU_0'/k is of the same order of magnitude as c_1 or (*b*) c_2 approximates c_1. The contingency (*a*) corresponds to Kelvin–Helmholtz or static instability and will not be considered further here (see Miles 1959*b*). The contingency (*b*) corresponds to resonance between the free-surface waves of the lower medium and the Tollmien–Schlichting waves associated with the shear flow in the upper medium. The latter contingency requires both the left-hand side and the denominator of the right-hand side of (6.1) to be expanded about either $c = c_1$ or $c = c_2$ to obtain a quadratic equation in either $c-c_1$ or $c-c_2$. An explicit exploration of this neighbourhood leads to rather involved algebraic expressions and requires

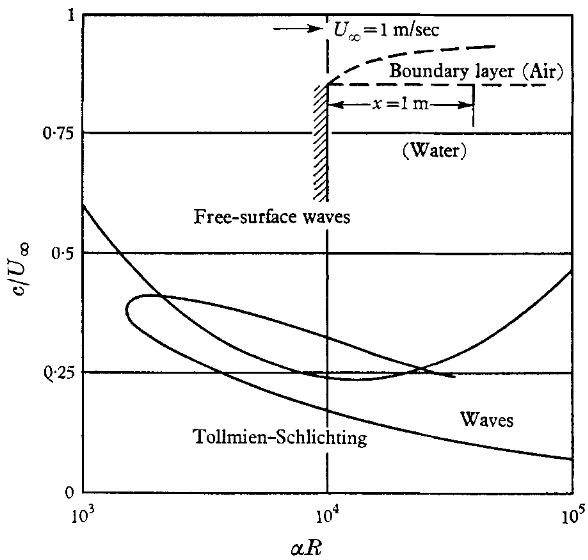

FIGURE 2. The neutral dispersion curve of Tollmien–Schlichting waves for a Blasius boundary layer compared with the neutral curve for deep-water waves; see (6.8) and (6.9).

the calculation of $\partial w/\partial c$, but it is evident that the interaction between the free-surface and Tollmien–Schlichting waves increases from $O(s)$ to $O(s^{\frac{1}{2}})$ as $c_1 - c_2 \to 0$.

We shall illustrate this last possibility by considering the known results for the stability of a Blasius boundary layer under zero pressure gradient. Using Lin's (1946) results for neutral Tollmien–Schlichting waves, we obtain the dispersion curve c/U_∞ *vs* αR of figure 2. In Lin's notation, α is a dimensionless wave-number based on the nominal thickness of the boundary layer, such that

$$\alpha R = 36kx, \tag{6.7}$$

where x is the distance from the leading edge of the boundary layer. We may compare this dispersion curve with that for deep-water waves, rewriting (2.14) in the form

$$\left(\frac{c_0}{U_\infty}\right)^2 = \left(\frac{36gx}{U_\infty^2}\right)(\alpha R)^{-1} + \left(\frac{\sigma}{36U_\infty^2 x}\right)\alpha R, \tag{6.8}$$

which is plotted in figure 2 for $g = 980$, $x = 100$, $U_\infty = 100$ and $\sigma = 73$, all in c.g.s. units. Increasing x would move this curve to the right (linearly with x on a linear scale, but αR has a logarithmic scale in figure 2), while increasing U_∞ would move it down. Allowing for finite depth also would move it down, but asymmetrically, tending to flatten the left-hand, or gravity wave, branch.

7. Growth of deep-water waves

We now consider the growth of surface waves on the basis of the approximation (6.5) together with the additional restriction $kd \gg 1$. Taking the imaginary part of (6.5) and multiplying through by k, we obtain the growth factor

$$\zeta = kc_i = \zeta_w + \zeta_a, \tag{7.1}$$

where

$$\zeta_w = -2k^2\nu_w - (2k^3\nu_w c_0)^{\frac{1}{2}} e^{-2kd} \tag{7.2}$$

and

$$\zeta_a = \tfrac{1}{2}sU_0'\mathscr{I}\left\{\frac{1+k\delta H(z)}{\mathscr{F}(z)-w(k,c)}\right\}_{c=c_0} \tag{7.3a}$$

$$\doteqdot \tfrac{1}{2}sU_0'\left[\frac{w_i-\mathscr{F}_i-k\delta(w_r-\mathscr{F}_r)\,H_i}{|\mathscr{F}-w|^2}\right]_{c=c_0}. \tag{7.3b}$$

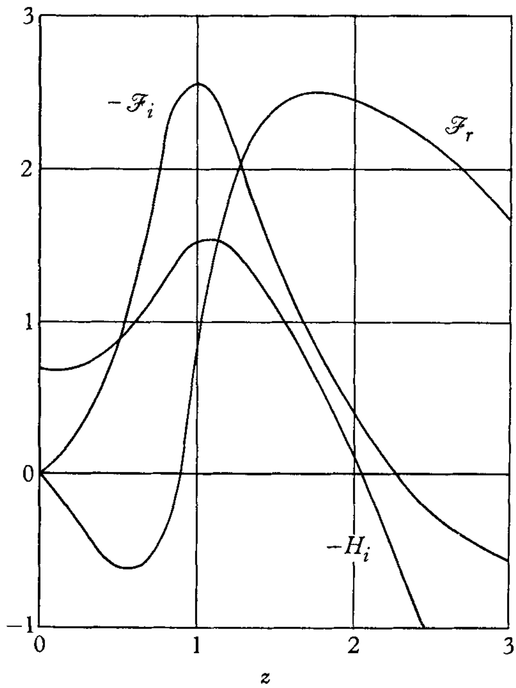

FIGURE 3. The functions $\mathscr{F}_r(z)$, $-\mathscr{F}_i(z)$ and $-H_i(z)$.

We have neglected $k\delta H_r$ compared with 1 but retained $k\delta H_i$ in (7.3*b*) against the contingency $|w_r-\mathscr{F}_r| \gg |w_i-\mathscr{F}_i|$. We also could neglect the second, or bottom-damping, term in (7.2) if $d \gg D$; see the discussion following (2.16).

The functions $\mathscr{F}_r(z)$, $-\mathscr{F}_i(z)$ and $H_i(z)$ are plotted in figure 3. As remarked originally by Brooke Benjamin (1959), the term $-\mathscr{F}_i(z)$ yields a positive energy transfer to the surface wave for $0 < z < 2{\cdot}3$. To this, we may add an additional energy transfer from the term $-H_i$ if $0 < z < 2{\cdot}1$ and $w_r > \mathscr{F}_r$ (the latter condition is likely to be satisfied for most configurations).

To focus attention on that range of wind speeds for which energy transfer through viscous phase shifts is significant, we introduce the friction velocity U_* according to

$$\nu_a U_0' = U_*^2 = \tau/s\rho, \tag{7.4}$$

where τ is the steady shear stress exerted by the shear flow $U(y)$ on $y = 0$. Substituting (7.4) into (3.9) and (3.4) and solving for U_*, we obtain

$$U_* = (2\pi\nu_a)^{\frac{1}{4}} z^{-\frac{3}{4}} \lambda^{-\frac{1}{4}} c_0^{\frac{3}{4}}, \tag{7.5}$$

where $\lambda = 2\pi/k$ denotes the wavelength. Assuming air over water ($\nu_a = 0{\cdot}154$, $g = 980$, $\sigma = 73$ c.g.s. units), U_* is plotted *vs* λ in figure 4 for $z = 1$ (corresponding to the maxima of $-\mathscr{F}_i$ and, very closely, $-H_i$) and $z = 2{\cdot}3$ ($\mathscr{F}_i = 0$ and $H_i \doteqdot 0$). The minimum of the lower curve occurs at $U_* = 4{\cdot}4$ cm/sec and $\lambda = 3{\cdot}8$ cm.

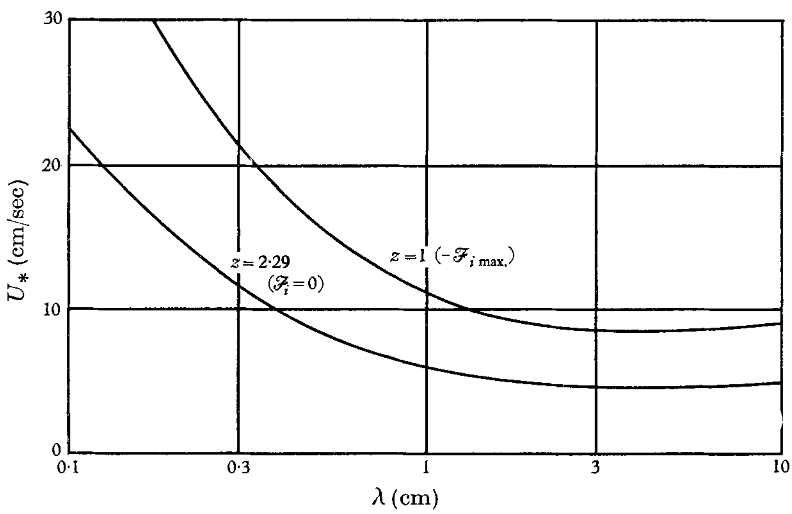

FIGURE 4. The lower curve gives the minimum wind speed for viscous energy transfer to deep-water gravity waves; the upper curve gives the order of magnitude of the wind speed for maximum, viscous energy transfer; see (7.5).

If U_* is appreciably in excess of the upper curve in figure 4, we may approximate $\mathscr{F}_i$ and H_i in (7.3) according to the leading terms of (4.8*a*) and (6.2*b*) and neglect w_i (which vanishes more rapidly than c/U_* as $z \to 0$) to obtain

$$\zeta = sU_0' w^{-2} (0{\cdot}322z + 0{\cdot}343k\delta w)$$

$$= skU_* \left[\frac{0{\cdot}322}{w^2} \left(\frac{U_*}{k\nu_a} \right)^{\frac{3}{3}} \left(\frac{c_0}{U_*} \right) + \frac{0{\cdot}343}{w} \left(\frac{U_*}{k\nu_a} \right)^{\frac{1}{3}} \right] \quad (z \ll 1,\ w_i \doteqdot 0). \tag{7.6}$$

We shall not be directly interested in the régime $z > 2{\cdot}3$, but we note in passing that substituting the asymptotic approximations (4.8*b*) and (6.2*c*) into (7.3) and retaining only the dominant terms yields

$$\zeta_a = \tfrac{1}{2} s U_0' \left\{ \frac{w_i - (\nu_a k/2c_0)^{\frac{1}{2}} [(U_c'/kc_0) + 2(w_r - 1)]}{(w_r - 1)^2 + w_i^2} \right\}. \tag{7.7}$$

This is in agreement with the writer's previous result (Miles 1959*a*, (4.7)) within the order of the approximations invoked there.

8. Numerical results for mean turbulent profile

We may apply the foregoing results to the generation of water waves by wind on the hypothesis that the mean velocity in the turbulent boundary layer may be regarded as a parallel shear flow, together with the hypotheses (5.4*a*, *b*). We also shall assume that the velocity profile is linear in $(0, \eta_1)$ according to

$$U(\eta) = (U_*^2/\nu_a)\,\eta, \quad 0 \leqslant \eta \leqslant \eta_1 \ll 1/k. \tag{8.1}$$

We then have $\gamma = w_i = 0$, and the energy transfer from shear flow to surface wave is associated entirely with the viscous Reynolds stress in the immediate neighbourhood of the air–water interface. We remark that the mean flow in this neighbourhood is approximately laminar, and hence that the neglect of the surface-wave induced perturbations of the turbulent Reynolds stresses in the boundary layer (see Miles 1957*a*) is relatively less important than in the corresponding calculation of the inviscid Reynolds stress. To be sure, these perturbations still are neglected in the calculation of w_r, but the success of the closely related model for Kelvin–Helmholtz instability (Miles 1959*b*) suggests that this is not likely to be too serious.

The differential equation (5.2) has been integrated numerically in $\eta > \eta_1$ for a velocity profile that fairs smoothly into that of (8.1) and is asymptotically logarithmic according to

$$U(\eta) \sim U_1 + \frac{U_*}{\kappa}\left[\log\left(\frac{4\kappa U_*\eta}{\nu_a}\right) - 1 + O\left(\frac{\eta_1}{\eta}\right)\right] \tag{8.2a}$$

$$= \frac{U_*}{\kappa}\log\left(\frac{U_*\eta}{\nu_a}\right) + C. \tag{8.2b}$$

This profile, based on a modification of Prandtl's mixing-length model for the turbulent boundary layer over a smooth rigid wall, agrees with observation if κ is taken to be 0·4 and U_1/U_* about 6·6 (as inferred from the experimental determination of C), but there is some evidence for larger values of U_1 for aerodynamically smooth flow over water (see the discussion in Miles 1957*b*).

The subroutine for this integration, together with a previously available subroutine for the determination of $\mathscr{F}(z)$, $G(z)$, and $H(z)$ through the integration of Airy's equation (Miles 1960*b*), has been used to determine ζ_a according to (7.3) for $\kappa = 0{\cdot}4$, $\nu_a = 0{\cdot}15$, $g = 980$, $\sigma = 73$, $U_1 = 5$ and 8, and various values of U_*, all in c.g.s. units. The results are presented in figures 5*a*, *b*. The deep-water $(kd \to \infty)$ damping term $\zeta_w = 2k^2\nu_w$ also is plotted for comparison; the actual growth rate for deep-water waves is proportional to the difference $\zeta_a - \zeta_w$. Damping curves for water of finite depth would lie above that shown as $-\zeta_w$ in figures 5*a*, *b* and could be calculated from (7.2) for $kd > 2$.

The aforementioned subroutine also was used to solve the eigenvalue equation (6.4). The implicit assumption $w_i = 0$ then requires $\mathscr{F}_i = 0$, which implies $z = 2{\cdot}3$ and $\mathscr{F}_r = 2{\cdot}3$, and we may reduce (6.4) to the solution of

$$w = \left(\frac{\kappa c}{U_*}\right) W\left[R_2, \kappa\left(\frac{U_1 - c_2}{U_*}\right)\right] = 2{\cdot}3 \tag{8.3a}$$

and $$R_2 = \kappa(U_*/k\nu_a) = \kappa(c_2/2{\cdot}3U_*)^3, \tag{8.3b}$$

where W is defined as in Miles (1962). The results for R_2 and c_2/U_* are plotted *vs* U_1/U_* in figure 6 for $\kappa = 0{\cdot}4$. Equating c_2 to $c_0(k)$, as given by (2.14) then yields a first approximation to the resonance condition between the natural oscillations

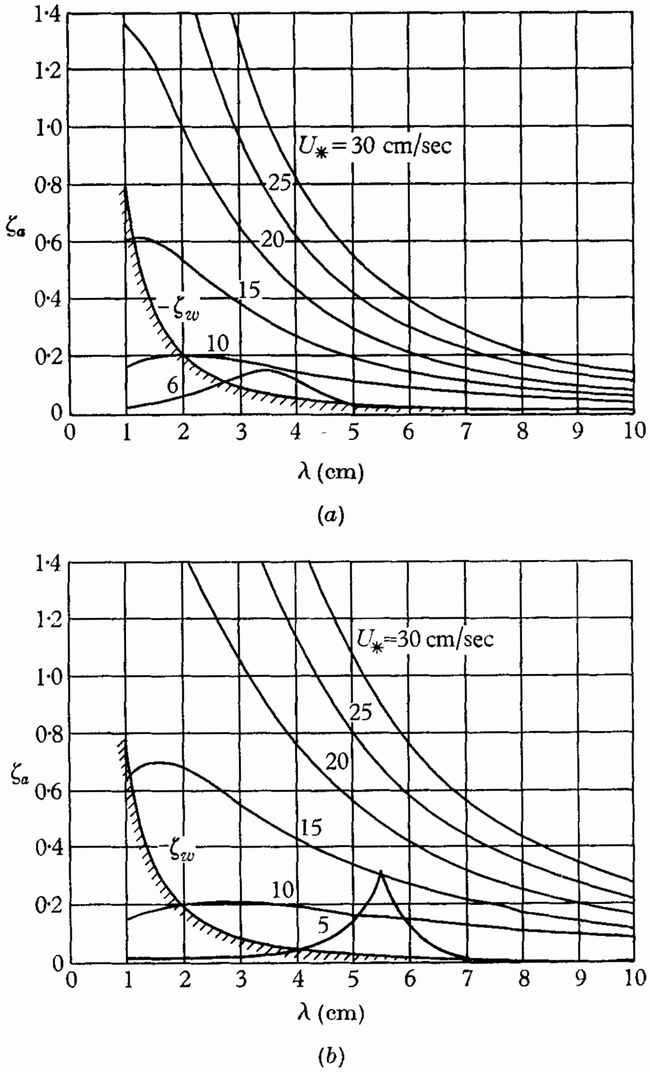

FIGURE 5. The growth rates for deep-water waves due to a turbulent wind having the mean velocity profile described by (8.1) and (8.2). (*a*) $U_1 = 5U_*$; (*b*) $U_1 = 8U_*$.

associated with the mean wind profile and deep-water waves on a free surface. The simultaneous values of λ and U_* implied by this resonance, assuming the previously listed numerical values of the various physical constants, are plotted *vs* U_1/U_* in figure 7. The minimum value of U_* is 4·4 cm/sec, in agreement with the lower curve of figure 4.

With one exception, the calculated peaks in the ζ_a *vs* λ curves of figures 5*a*, *b* appear to be closely associated with the realization of the maximum possible

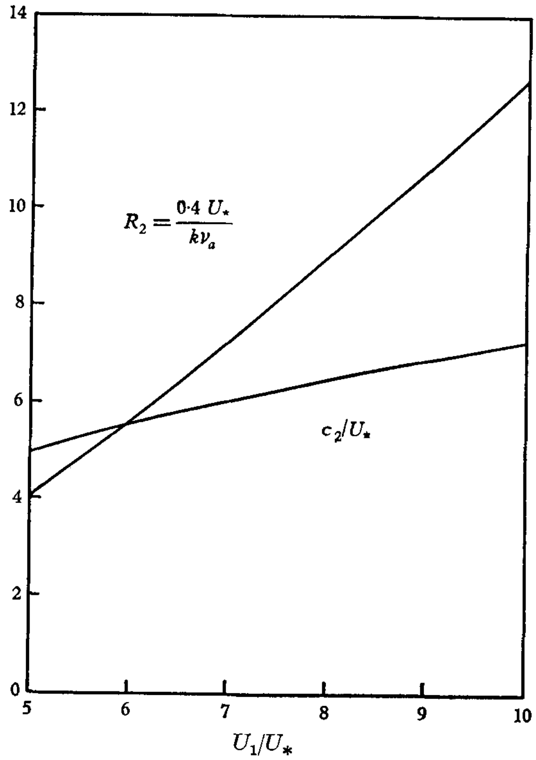

FIGURE 6. The solution of (6.4) for the mean velocity profile described by (8.1) and (8.2).

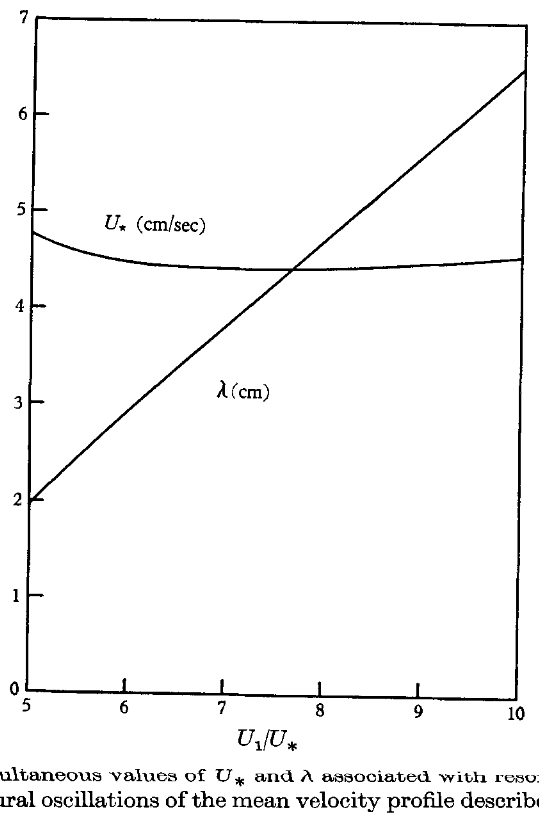

FIGURE 7. The simultaneous values of U_* and λ associated with resonance of deep-water waves with the natural oscillations of the mean velocity profile described by (8.1) and (8.2).

values of $-\mathscr{F}_i(z)$ for the specified values of U_*. The peak at $\lambda = 5{\cdot}5$ cm and $U_* = 5$ cm/sec in figure 5*b* closely approximates the resonant point of $\lambda = 4{\cdot}7$ cm and $U_* = 4{\cdot}5$ cm/sec, however, and therefore appears to be a resonant peak in the sense of the preceding discussion. It also appears likely that there exists a rather narrow range of U_*, for given U_1/U_*, in which the ζ_a *vs* λ curves would exhibit much stronger and sharper peaks than those shown in figure 5, in consequence of which the results presented there may be misleading. The writer hopes to explore this conjecture further.

The peak value of ζ_a in figure 5*b* for $U_* = 4{\cdot}5$ cm/sec is found to be 0·84 at $\lambda = 4{\cdot}7$ cm. This implies that the amplitude of 4·7 cm waves actually will decrease as the windspeed is increased in the neighbourhood of $U_* = 4{\cdot}5$ cm/sec if $U = 8U_*$; but of course this does not imply that the r.m.s. amplitude of the total spectrum would decrease.

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